INVESTIGATION OF THE STABILITY OF MOTION OF A COMPRESSIBLE GAS

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Some recent investigations have been concerned with stability problems of the motion of a gas of constant density in its own gravitational field [1–3]. The same methods can be used to study the behavior of a spherically symmetric mass of compressible gas with a spatially constant density in the absence of gravity.

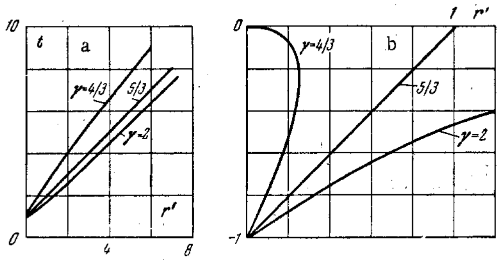

Fig. 1

As the principal parameters of motion we consider the density ρ and velocity $\mathbf{u}$:

$$\rho = \frac{\rho_0}{|t|^3}, \quad \mathbf{u} = \frac{\mathbf{r}}{t},$$

where $\mathbf{r}$ is the Euler radius, ρ_0 is a constant, $t \to +\infty$ stands for expansion and $t \to -\infty$ for compression. This motion corresponds to omnidirectional expansion or compresion; the motion remains spherically symmetrical with respect to any point in space. In the following, we consider the motion to be adiabatic, with an adiabatic exponent $2 > \gamma > 1$ (real gas).

1. It seems appropriate first to discuss qualitatively the causes of onset of instability. In the presence of gravity, the principal cause of instability is the gravity force; both pressure and viscosity are only capable of weakening instability (in Newtonian theory).

In the absence of gravity, the cause of instability is kinematics. It appears that any motion of a compressible substance with a spatially constant density

$$\rho = \frac{\text{const}}{t^\alpha} \qquad (\alpha > 0)$$

possesses the following peculiarity: if the same motion occurs in a separate region of space, but a time lead is involved

$$\rho' = \frac{\text{const}}{|t + \Delta|^\alpha} \qquad (\Delta > 0,\ |t| > \Delta)$$

(Δ is small except in the region of joint solutions), then

$$\frac{\rho' - \rho}{\rho} = -\frac{\Delta}{t}\alpha.$$

For compression ($t \to -0$) the solution involving a time lead begins to depart rapidly from the basic one. Considering the time-lead solution as a disturbance imposed on the basic solution (such disturbances have been examined by Zel'dovich [3]), its solution will depend to a great extent on the equation of state.

In qualitative terms, this is explained by the fact that the problem contains two velocities: Euler's velocity $u = r/t$, that depends on the radius, and the speed of sound c that is constant in space.

$$c \sim \rho^{1/2(\gamma-1)}, \qquad c = c_0 t^{-3/2(\gamma-1)}, \qquad c_0 = \text{const}. \tag{1.1}$$

If at any point in space there is a spherically symmetrical disturbance, the center of this disturbance can always be taken as the origin of the system of coordinates. If the initial dimensions of the disturbance are not considered, then in the spreading of the disturbance in the r, t plane, the trajectory of its boundary will be the curve

$$r = \pm \frac{c_0}{3/2(\gamma-1)}\left[|t|^{3/2(5/3-\gamma)} - At\right] \qquad (A = \text{const}). \tag{1.2}$$

Here, the plus and minus signs correspond to compression and expansion, respectively. From this example and the figure (where $r' = 3/2(\gamma - 1)c_0^{-1}r$), it can be seen that at $\gamma < 5/3$ the compression is unstable; on the other hand, the expansion is of such a nature that as $t \to \infty$, the disturbance is spread over a finite mass ($M \sim r^3/t^3$), i.e., $\rho^{-1}\delta\rho \to$ const.

2. Let us make a more exact analysis. The disturbed motion is represented in the form

$$\rho = \frac{\rho_0}{|t|^3}[1 + \omega(r, t)], \qquad \mathbf{u} = \frac{\mathbf{r}}{t}[1 + V(r, t)]. \tag{2.1}$$

Introducing Lagrange coordinates (mass $\sim \rho r^3 \sim r^3/t^3$, $R = r/t$), the continuity equation and the Navier–Stokes equation can be readily written in the form

$$t\frac{\partial\omega}{\partial t} + \frac{1}{R^2}\frac{\partial}{\partial R}(VR^3) = 0,$$

$$t\frac{\partial V}{\partial t} + V = -\frac{c^2}{R}\frac{\partial\omega}{\partial R} + \frac{4}{3}\eta\frac{|t|^3}{\rho_0 t}\left[\frac{\partial^2 V}{\partial R^2} + \frac{4}{R}\frac{\partial V}{\partial R}\right] \tag{2.2}$$

The first equation indicates that if the disturbance has not yet appeared at radius R_1, then

$$\int_0^{R_1} \omega R^2 dR = \text{const}. \tag{2.3}$$

Combining the equations (2.2) and (2.3), it is possible to obtain an equation for $\omega(R, t)$. We will develop the solution of this equation in a series (or an integral, depending on the boundary conditions)

$$\omega(R, t) = \sum_k \omega(kt) \frac{\sin kR}{kR}.$$

The function $\omega(k, t)$ satisfies the equation

$$t^2 \frac{\partial^2 \omega}{\partial t^2} + 2t \left(1 + \frac{2}{3} \eta \frac{|t|^3}{\rho_0 t} k^2\right) \frac{\partial \omega}{\partial t} + c^2 k^2 \omega = 0. \qquad (2.4)$$

The viscosity of the gas is proportional to $\sqrt{T}$, where T is the temperature, i. e.,

$$\eta \sim T^{1/2} \sim \rho^{1/2(\gamma-1)} \sim t^{-3/2(\gamma-1)}.$$

It is obvious that for any wavelength, in compression, there is a moment of time, starting from which viscosity may be neglected, while in the case of expansion, viscosity may be neglected at moments that are close to the onset of disturbance. We may, therefore, examine the motion without considering viscosity. Then the equation reduces to the Bessel equation ($c^2 = c_0{}^2 |t|^{-3(\gamma-1)}$. Its solution (for $\gamma \neq 1$) is

$$\omega(k, t) = |t|^{1/2} \{C_1 J_\nu(x) + C_2 J_{-\nu}(x)\},$$

where

$$\nu = \frac{1}{3(\gamma-1)}, \qquad x = \frac{2c_0|k|}{3(\gamma-1)} |t|^{-3/2(\gamma-1)} \quad \text{for } t \to -0, \qquad (2.5)$$

$$\omega(k, t) \to |t|^{3/4(\gamma-5/3)} \cos[x \mp 1/2\pi\nu - 1/4\pi]. \qquad (2.6)$$

Hence for compression, it can be seen that the motion is unstable if $\gamma < 5/3$. The amplitude of the standing wave increases in an oscillatory fashion. If $\gamma > 5/3$, the motion is stable. The amplitude of the standing wave decreases.

In the case of expansion, at a moment close to the onset of disturbance (t increasing, but still close to +0), we have the reversed pattern. The motion is stable for $\gamma < 5/3$ and unstable for $\gamma > 5/3$. For expansion, when $t \to +\infty$,

$$\omega(k, t) \to \text{const}, \qquad (2.7)$$

the disturbance tends toward a finite value. Thus, the results of an exact analysis fully support the qualitative analysis.

The motion may be unstable. Instability is defined by the equation of state, and is independent of the wavelength (for $\gamma \neq 1$).

In the isothermal case ($\gamma = 1$), if viscosity is neglected, the equation (2.4) has the solution

$$\omega(k, t) = C_1 t^{\alpha_1} + C_2 t^{\alpha_2}, \qquad \alpha_{1,2} = 1/2 \pm \sqrt{1/4 - C^2 - k^2},$$

where C_1 and C_2 are constants; when α is complex, $C_1 = C_2^*$. In the isothermal case, the motion is unsteady and, contrary to the general case, the increase in amplitude depends on the wavelength.

In all cases, for expansion at large intervals of time the influence of viscosity becomes apparent, which ensures attenuation, as can be seen from (2.4).

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